Using fuzzy DEA to assess efficiency in education: An application to American schools

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Abstract

Many studies devoted to efficiency performance evaluation in the education sector are based on data aggregated at school level (measures of central tendency) calculated from the average values of students belonging to the same school. Although this is a common and accepted way of summarizing data from the original observations (students), it is not less true that this approach neglects the existing dispersion of data, which may become a serious problem if variability across schools is high. This paper uses data from PISA 2012 for the United States to show that efficiency evaluations of schools based on aggregate data may not reflect appropriately what happens within schools. In order to operationalize our approach, we resort to fuzzy Data Envelopment Analysis since this methodology allows dealing with imprecise data and the notion of fuzziness in some variables such as the socio-economic status of students or test scores.

Keywords: Education, Efficiency, Data Envelopment Analysis, Fuzzy Data.

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1. Introduction

Since the pioneering works by Bessent and Bessent (1980), Charnes et al. (1981) and Bessent et al. (1982), studies focused on measuring efficiency in the education sector have become extremely popular (see Johnes, 2015 or De Witte and López-Torres, 2017 for recent reviews of this literature). In this sense, the actual constraints of resources faced by most of countries and the great amount of national income devoted to education expenditures have lead policy makers and researchers to become increasingly concerned with the assessment of efficiency within this field.

In order to establish the existing relationship between inputs and outputs, empirical studies usually rely on the so-called educational production function (Levin, 1974; Hanushek, 1979), in which the common inputs are things like school resources, teacher quality students´ background and the outcome is student achievement typically measured by test scores. Since the evaluated units are frequently schools, districts or states/regions, data about input and output variables needs to be aggregated by calculating an average value. Although this a common practices followed by many researchers in the education field, it is worth mentioning that this procedure raises a number of further considerations that we attempt to explore in the present work.

Data Envelopment Analysis (DEA) is a non-parametric methodology for estimating technical efficiency of a set of Decision Making Units (DMUs) from a dataset of inputs and outputs. This methodology is fundamentally based on Mathematical Programming and allows a piece-wise linear production frontier enveloping the input-output observations to be determined. Although, in its origins, DEA was limited to cope with 'crisp' data, i.e. precise measurements on inputs and outputs of the observations, some authors have proposed various methods for dealing with imprecise and vague data in DEA. Without the aim to be exhaustive, Sengupta (1987) generalized the DEA for measuring efficiency of DMUs in the presence of stochastic variations of input and output data, considering three types of data variations (in the objective function, in the constraints and the outputs). In this same line, Land et al. (1993), Olesen and Petersen (1995) and Cooper et al. (1996) applied the chance constrained programming to DEA models, providing a way of enveloping confidence regions for observed stochastic multiple-input and multiple output observations. Cooper et al. (1999) introduced the Imprecise Data Envelopment Analysis (IDEA) method for mixing imprecisely and exactly known information with the objective of estimate technical efficiency when the data may be known only within specified bounds or in terms of ordinal relations. Bruni et al. (2009) proposed stochastic DEA model based upon the theory of joint probabilistic constraints, which can be easily used with general multivariate distribution functions.

Another approach, which will be the one on which we based our study, is the Fuzzy DEA. Fuzzy DEA deals with imprecise data modeled through the notion of fuzziness (Zadeh, 1975) and usually consists of transforming the fuzzy DEA model into several conventional ‘crisp’ DEA programs. Some papers devoted to FDEA are Carlsson and Korhonen (1986), Sengupta (1992), Triantis and Girod (1998), Kao and Liu (2000) and, more recently, Tavana et al. (2012), Saati et al. (2013) and Hatami-Marbini et al. (2017), to name but a few since this has been a very productive field of research in DEA.

In particular, this paper applies the methodology proposed by Kao and Liu (2000) to show the effects of considering the variability of the data (inputs and outputs), since it is the most cited and applied contribution on FDEA following Emrouznejad et al. (2014). We will use this technique to measure the efficiencies of a sample of American schools participating in PISA 2012 with some fuzzy variables: the socio-economic status of students (input) and their test scores in reading and maths (outputs). The idea of the Kao and Liu (2000) approach is to transform the corresponding fuzzy ‘radial’ DEA model to a family of conventional crisp radial DEA models by applying the α-cut procedure. Since the efficiency measures are expressed by membership functions rather than directly by crisp values (the means of the variables), more information will be provided from the point of view of management.

The remainder of the paper is organized as follows: In Section 2, we introduce the necessary notation and background. Subsequently, in Section 3, we show and discuss the main results of the application of the Kao and Liu (2000) approach on the data from the American schools and its comparison with the standard radial DEA model. In Section 4, we present the conclusions.

2. Methodology

In this section, we briefly review the traditional radial DEA models as well as the Kao and Liu approach. Additionally, we discuss on the used variables and the data source.

2.1. Notation, background and the Kao and Liu approach

Before presenting some basic notions and models of DEA, we first of all need to introduce some notation.

Let us consider *n* DMUs (e.g., schools in our context) to be evaluated. DMU$_j$ consumes $x_j = \left(x_{1j},...,x_{mj}\right) \in R_+^m$, $x_j \neq 0_m$, amounts of inputs for the production of $y_j = \left(y_{1j},...,y_{sj}\right) \in R_+^s$, $y_j \neq 0_s$, amounts of outputs. The relative efficiency of each DMU in the sample is assessed with reference to the so-called production possibility set, which can be non-parametrically constructed from the observations by assuming certain postulates (see Banker et al., 1984). In this way, the production possibility set in DEA, *T,* can then be mathematically characterized under Variable Returns to Scale (VRS) as follows:

$$T_{VRS} = \left\{ \left(x,y\right) \in R_+^m \times R_+^s : x \geq \sum_{j=1}^{n} \lambda_j x_j, y \leq \sum_{j=1}^{n} \lambda_j y_j, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, j = 1,...,n \right\}. \quad (1)$$

Regarding the measurement of technical efficiency, the two usual approaches are linked to the input and output orientations. Seeking simplicity, we focus our revision on the output-oriented approach, which additionally will be the selected perspective in our application. Nevertheless, a similar analysis could be performed in the case of input orientation.

Output-oriented models assume that each DMU is interested in maximizing outputs while using no more than the observed amount of any input. The first introduced output-oriented model in DEA was the so-called radial model, which must be solved for each DMU in the data sample, and that increases equi-proportionally the mix of produced outputs of the evaluated DMU while keeping constant the input levels of this same unit. These changes are assessed with respect to the (estimated) production possibility set T_{VRS} in (1). The output-oriented radial DEA model under VRS to assess the performance of DMU$_0$ with vector of inputs and outputs $\left(x_0, y_0\right)$ may be formulated as:

$$
\begin{aligned}
&Max \quad \phi_0 \\
&s.t. \\
&\qquad \sum_{j=1}^{n} \lambda_{j0} x_{ij} \le x_{i0}, \qquad i = 1,\dots,m \qquad (2.1) \\
&\qquad \sum_{j=1}^{n} \lambda_{j0} y_{rj} \ge \phi_0 y_{r0}, \quad r = 1,\dots,s \qquad (2.2) \\
&\qquad \sum_{j=1}^{n} \lambda_{j0} = 1, \qquad\qquad\qquad\qquad (2.3) \\
&\qquad \lambda_{j0} \ge 0, \qquad\qquad j = 1,\dots,n \qquad (2.4)
\end{aligned}
\qquad (2)
$$

The optimal value ϕ_0^* of model (2), which is a linear program, is always greater or equal to one. Indeed, ϕ_0^* is baptized as the efficiency score of DMU_0. Following the Debreu-Farrell definition of technical efficiency, if $\phi_0^* = 1$, then DMU_0 is efficient in the sense that it is not possible to improve equi-proportionaly all its outputs without increasing some of its inputs. Graphically, the point y_0 is located onto the isoquant of the production possibility set given x_0. Otherwise, if $\phi_0^* > 1$, then DMU_0 is inefficient and it has room for increasing its outputs consuming the original quantity of inputs.

Given the linear nature of program (2), by duality in Linear Programming, it is also possible to determine the efficiency score of DMU_0 by solving (3).

$$
\begin{aligned}
&Min \quad \sum_{i=1}^{m} v_{i0} x_{i0} - \pi_0 \\
&s.t. \\
&\qquad \sum_{r=1}^{s} u_{r0} y_{r0} = 1, \qquad\qquad\qquad\qquad\qquad (3.1) \\
&\qquad \sum_{i=1}^{m} v_{i0} x_{ij} - \sum_{r=1}^{s} u_{r0} y_{rj} - \pi_0 \ge 0, \quad j = 1,\dots,n \qquad (3.2) \\
&\qquad v_{i0} \ge 0, \qquad\qquad\qquad i = 1,\dots,m \qquad (3.3) \\
&\qquad u_{r0} \ge 0, \qquad\qquad\qquad r = 1,\dots,s \qquad (3.4)
\end{aligned}
\qquad (3)
$$

In the above exposition, we have assumed that all information collected on inputs and outputs of the assessed units is precise (crisp). However, the observed values of the variables (inputs and outputs) in real-world problems are sometimes imprecise or rough. In our application, for example, the results in the test of reading is seen in the specialized literature as a proxy of the real outputs of the school. However, although the production unit in this context is the school, we do not have a unique value for the test score in reading. Instead, we have a sample of students of this school that have taken the test and, consequently, we have a set of scores that usually is aggregated through the arithmetic mean of the data. This mean is then utilized as the value in this particular output for the corresponding school (DMU). So, this is the numerical value that we should use in model (2), along with the rest of the sample data in order to solve the 'crisp' traditional DEA radial model. Unfortunately, the mean cannot be a suitable single representative value for the data sample, especially if the spread or variation of the students' scores in the reading test is high. It seems natural that two schools, A and B, with the same mean in their variables but with a dissimilar pattern with respect to the variability in the data are classified in a different way regarding technical efficiency.

As we aforementioned, a way of incorporating this data variability in the efficiency analysis in DEA is by means of the application of Fuzzy Data Envelopment Analysis, where variables as the score in the reading test is included in the DEA model as a fuzzy number instead of a crisp number. Emrouznejad et al. (2014) provide a taxonomy and review of the FDEA methods existing in the literature, with six categories: the tolerance approach, the α -level based approach, the fuzzy ranking approach, the possibility approach, the fuzzy arithmetic and the fuzzy random/type-2 fuzzy set. Among them, the α -level based approach is probably the most popular FDEA model and Kao and Liu (2000), which belongs to this category, the most cited and applied methodology. Consequently, we next revise the main characteristics of this last approach.

The basic idea behind the Kao and Liu approach is to apply the α -cuts and Zadeh's extension principle to transform a fuzzy DEA model to a series of conventional crisp DEA models. These conventional models are then solved by well-known Linear Programming techniques. In particular, Kao and Liu exploit the linear model (3).

Let us suppose that the inputs $\tilde{x}_{ij}$, $j=1,...,n$, $i=1,...,m$, and outputs $\tilde{y}_{rj}$, $j=1,...,n$, $r=1,...,s$, are fuzzy numbers with membership functions $\mu_{\tilde{x}_{ij}}$ and $\mu_{\tilde{y}_{rj}}$, respectively. The membership functions characterizes the fuzzy numbers and measures the degree of truth of the statement that the represented magnitude takes a specific value (e.g., input #1 for DMU_0 takes value 5.7). For example, a trapezoidal fuzzy output $\tilde{y}_{rj}$ (see Figure 1) could be defined from:

$$\mu_{\tilde{y}_{rj}}(z)=\begin{cases} z-5, & 5\le z\le 6 \\ 1, & 6\le z\le 8 \\ 9-z & 8\le z\le 9 \end{cases} \quad (4)$$

<Please insert Figure 1 approximately here>

A usual notion in fuzzy number theory is the α-cuts, also called α-possibility level sets, of the membership function. Let $\left(\tilde{x}_{ij}\right)_{\alpha}=\left\{z:\mu_{\tilde{x}_{ij}}(z)\ge\alpha\right\}$ and $\left(\tilde{y}_{rj}\right)_{\alpha}=\left\{z:\mu_{\tilde{y}_{rj}}(z)\ge\alpha\right\}$ be the α-cut of $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$, respectively. Each α-cut generates a confidence interval for the represented magnitude (input or output). This concept is also illustrated in Figure 1. In particular, for the specific possibility level $\alpha=0.5$ we get the interval $\left(\tilde{y}_{rj}\right)_{0.5}=\left\{z:\mu_{\tilde{y}_{rj}}(z)\ge 0.5\right\}=[5.5,8.5]$. In general form, we will represent the α-cuts by means of its lowest and highest values in the generated interval: $\left(\tilde{x}_{ij}\right)_{\alpha}=\left[\left(\tilde{x}_{ij}\right)_{\alpha}^{L},\left(\tilde{x}_{ij}\right)_{\alpha}^{U}\right]$ and $\left(\tilde{y}_{rj}\right)_{\alpha}=\left[\left(\tilde{y}_{rj}\right)_{\alpha}^{L},\left(\tilde{y}_{rj}\right)_{\alpha}^{U}\right]$.

Without loss of generality, it is usual assume that all input and output data for all DMUs are fuzzy numbers, since crisp values may be represented by degenerated membership functions that only have one value in their domain.

In this way, the radial DEA model (3) expressed by means of the input and output fuzzy numbers would be formulated as follows.

$$\begin{aligned} \tilde{\phi}_0^{*} = \quad & Min \quad \sum_{i=1}^{m} v_{i0}\tilde{x}_{i0}-\pi_0 \\ & s.t. \\ & \sum_{r=1}^{s} u_{r0}\tilde{y}_{r0}=1, && (5.1) \\ & \sum_{i=1}^{m} v_{i0}\tilde{x}_{ij}-\sum_{r=1}^{s} u_{r0}\tilde{y}_{rj}-\pi_0\ge 0, \quad j=1,\ldots,n && (5.2) \\ & v_{i0}\ge 0, \quad i=1,\ldots,m && (5.3) \\ & u_{r0}\ge 0, \quad r=1,\ldots,s && (5.4) \end{aligned} \quad , \quad (5)$$

where $\tilde{\phi}_0^{*}$ represents the efficiency score for DMU_0 and that is also a fuzzy number with a certain membership function. The Kao and Liu approach is based on determining this membership function from different α-cuts of the fuzzy numbers that appear in model (5). Indeed, given a certain possibility level α ($0 < \alpha \leq 1$), it is possible to determine the lowest value of the corresponding α-cut for the membership function of $\tilde{\phi}_0^*$ through (6).

$$\left(\tilde{\phi}_0^*\right)_\alpha^L = Min \quad \sum_{i=1}^{m} v_{i0}\left(\tilde{x}_{i0}\right)_\alpha^U - \pi_0$$

$$s.t.$$

$$\sum_{r=1}^{s} u_{r0}\left(\tilde{y}_{r0}\right)_\alpha^L = 1, \qquad (6.1)$$

$$\sum_{i=1}^{m} v_{i0}\left(\tilde{x}_{i0}\right)_\alpha^L - \sum_{r=1}^{s} u_{r0}\left(\tilde{y}_{r0}\right)_\alpha^U - \pi_0 \geq 0, \quad j \neq 0 \qquad (6.2) \quad , \qquad (6)$$

$$\sum_{i=1}^{m} v_{i0}\left(\tilde{x}_{i0}\right)_\alpha^U - \sum_{r=1}^{s} u_{r0}\left(\tilde{y}_{r0}\right)_\alpha^L - \pi_0 \geq 0, \qquad (6.3)$$

$$v_{i0} \geq 0, \qquad i = 1,\ldots,m \qquad (6.4)$$

$$u_{r0} \geq 0, \qquad r = 1,\ldots,s \qquad (6.5)$$

In (6) we use the idea that to calculate the smallest efficiency score of DMU_0 compared with the other *n*-1 DMUs, one must set the output level of DMU_0 and the input levels of all other units to their lowest values and set the input level of DMU_0 and the output levels of all other DMUs to their highest values.

By analogy, it is possible to calculate the highest value of the corresponding α-cut for the membership function of $\tilde{\phi}_0^*$ as follows.

$$\left(\tilde{\phi}_0^*\right)_\alpha^U = Min \quad \sum_{i=1}^{m} v_{i0}\left(\tilde{x}_{i0}\right)_\alpha^L - \pi_0$$

$$s.t.$$

$$\sum_{r=1}^{s} u_{r0}\left(\tilde{y}_{r0}\right)_\alpha^U = 1, \qquad (7.1)$$

$$\sum_{i=1}^{m} v_{i0}\left(\tilde{x}_{i0}\right)_\alpha^U - \sum_{r=1}^{s} u_{r0}\left(\tilde{y}_{r0}\right)_\alpha^L - \pi_0 \geq 0, \quad j \neq 0 \qquad (7.2) \quad , \qquad (7)$$

$$\sum_{i=1}^{m} v_{i0}\left(\tilde{x}_{i0}\right)_\alpha^L - \sum_{r=1}^{s} u_{r0}\left(\tilde{y}_{r0}\right)_\alpha^U - \pi_0 \geq 0, \qquad (7.3)$$

$$v_{i0} \geq 0, \qquad i = 1,\ldots,m \qquad (7.4)$$

$$u_{r0} \geq 0, \qquad r = 1,\ldots,s \qquad (7.5)$$

Programs (6) and (7) permit the systematic study of the form of the membership function of the 'fuzzy' efficiency score of DMU_0 simply by determining the interval $\left[\left(\tilde{\phi}_0^*\right)_\alpha^L, \left(\tilde{\phi}_0^*\right)_\alpha^U\right]$ for different values of α $(0 < \alpha \leq 1)$.

After the fuzzy efficiency scores are determined for all DMUs in the sample, an interesting procedure is to rank the units to determine the better ones. Although there are several methods for ranking units in FDEA, we will resort to that suggested by Kao and Liu (2000) since it is based upon the α-cuts. The proposed index is defined as:

$$I_0 = \left[\sum_{k=0}^{h}\left(\left(\tilde{\phi}_0^*\right)_{\alpha_k}^U - c\right)\right] \Big/ \left[\sum_{k=0}^{h}\left(\left(\tilde{\phi}_0^*\right)_{\alpha_k}^U - c\right) - \sum_{k=0}^{h}\left(\left(\tilde{\phi}_0^*\right)_{\alpha_k}^L - d\right)\right], \tag{8}$$

where $c = \min_{j,k}\left\{\left(\tilde{\phi}_j^*\right)_{\alpha_k}^L\right\}$ and $d = \max_{j,k}\left\{\left(\tilde{\phi}_j^*\right)_{\alpha_k}^U\right\}$.

2.2. Variables and data

We use data for from the PISA (Programme for International Student Assessment) 2012 survey for the United States.. This dataset provides results on the performance of 15 year-old students in different competences as well as other factors potentially related to those results such as variables representing student background, school environment or educational provision.

Following the well-established literature on school efficiency (e.g. Agasisti and Zoido, 2015; De Witte and Lopez-Torres, 2015; Santin and Sicilia, 2015; Crespo-Cebada et al., 2014), we select the results from a standardized test as educational outputs and three usual inputs in education production functions such as the students (raw material), infrastructures (school resources) and teachers (human capital). A detailed explanation of the specific indicators considered in the empirical analysis is provided below.

- As a proxy for the quality of students in the school, we use the socio-economic status of students in the school, represented by the ESCS index, which provides a measure of family background that includes the highest levels of parents´ occupation, educational resources and cultural possessions at home. Since the original values of this variable presented positive and negative values, all of them were rescaled to show positive values.

- As a proxy for the availability of material resources, we use an index created by PISA analysts (SCMATEDU) from the responses given by school principals regarding several educational resources such as computers, educational software, calculators, books, audio visual resources or laboratory equipment. In this case, we have also rescaled the original values to assure that all values are positive.

- The inverse of the student-teacher ratio, i.e., the number of teachers per (hundred) students (TEACHERS), as a proxy for human resources employed by schools.

Variables ESCS, SCMATEDU and TEACHERS are considered inputs, whereas variables MATH and READING are considered outputs in the analysis. While SCMATEDU and TEACHERS can be treated as crisp, in the sense that they are measured without intra-school variability, we cannot claim the same for MATH, READING and ESCS, which come from surveys with multiple individuals (students). We will deal with these three variables as fuzzy.

The original dataset consisted of 162 schools with a total of 4,978 students. Each school presented a different number of students (from 1 to 40 scholars). So, in order to have a sample of schools with a large enough number of individuals, we deleted from the analysis those schools with a number of responses less than 20. The application of this filter implied the elimination of 17 schools. Therefore, the final database consisted of 145 schools and 4,797 pupils.

The traditional efficiency analysis of data from PISA would deal with ESCS, MATH and TEACHERS as crisp variables and, as usual, these three variables would be represented by the averaged responses from the students belonging to the same school, neglecting the intra-school variability in the responses. In particular, Table 1 reports the descriptive statistics for the five used variables in our study under the traditional perspective.

<Please insert Table 1 approximately here>

If we compare the standard deviations of Table 1 with those determined using the microdata, i.e. the students' responses in each school, we get very different figures. In particular, for MATH, the mean of the standard deviations of the responses in the test in each school equal to 76.28, which is very far from 44.64 (Table 1). For READING, we get a mean of the standard deviations equals 77.73, which contrasts to 45.92 in the table. Finally, regarding the fuzzy input ESCS, we have 0.8 vs 0.53. Again, the result is very different from that obtained under the usual approach.

However, the variables MATH, READING and ESCS present intra-school variability. For example, Table 2 shows the mean and the standard deviation corresponding to school #115 for these three variables. Note that ESCS, in particular, presents a variation coefficient of 38%, which is high, indicating that the responses in this school are not quite well represented by their mean (2.52).

<Please insert Table 2 approximately here>

In order to apply the approach by Kao and Liu (2000), it is necessary to model each value of the variables MATH, READING and ESCS for each school as a particular fuzzy number. To do that, in our case, we opted for estimating a kernel function from the data corresponding to each variable for each school. In particular, we used the function 'density' from the R package 'stats' (R Core Team, 2016). For example, for the school #15, the kernel corresponding to variable MATH appears graphically represented in Figure 2.

<Please insert Figure 2 approximately here>

We additionally calculate the skewness coefficient for each kernel, obtaining values close to zero, indicating that the kernel functions are quite symmetrical (see Table 3). This implies that the mean value of the distributions are very similar to the median and mode.

<Please insert Table 3 approximately here>

Following the Kao and Liu procedure, given a certain possibility level α $(0 < \alpha \leq 1)$, we need to determine the α-cut (an interval, which can be described by means of its lowest and highest values). To do that, we first need to rescale the values of the y-axis in the corresponding kernel in such a way that the maximum kernel function takes value 1 in the new y-axis. In this way, we can consider different values of α $(0 < \alpha \leq 1)$ and get the α-cut. For example, for school #115, α equals to 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 and 1 generates the confidence intervals that we show in Table 4. Note that $\alpha = 1$ yields an interval consisting of a unique point, that corresponding to the mode of the kernel function.

<Please insert Table 4 approximately here>

Once we determine all the necessary α-cuts for all the variables and schools, we apply the Kao and Liu programs, (6) and (7), to estimate the membership function of the fuzzy efficiency score of each DMU in the sample. The main results are shown in the next section.

3. Empirical application

Table 5 shows a summary of the results obtained with the approach by Kao and Liu (2000). Particularly, we have shown the α-cuts corresponding to the fuzzy efficiency score of each school for $\alpha = 0.5, 0.6, 0.7, 0.8, 0.9$ and 1. Additionally, we have also determined both the traditional efficiency score through model (2), based on the mean values of ESCS, MATH and READING, and index (8), which allows to summarize the different results determined for the set of α's. For each α, we show two associated columns in the table. One represents the lowest value of the confidence interval related to the α-cut, whereas the other one represents the highest value for the fuzzy efficiency score $\tilde{\phi}_0^*$. In other words, the columns represent the interval $\left[\left(\tilde{\phi}_0^*\right)_\alpha^L, \left(\tilde{\phi}_0^*\right)_\alpha^U\right]$.

<Please insert Table 5 approximately here>

From our results, we observe that the scores for the traditional model and for $\alpha = 1$ are very similar. Indeed, they present a correlation value of 0.86. This is consequence of two facts. First, considering $\alpha = 1$ yields the mode of the data for each variable and school as the corresponding α-cut. Second, Table 3 showed, through the skewness coefficient, that modes are close in general to the means of the data in this real example due to the symmetry of the data distribution. Consequently, the results of programs (6) and (7) for $\alpha = 1$ are identical and generate an optimal value (score) very similar to the optimal value of models (2) and (3), where ESCS, MATH and READING are represented by the mean values in each school. Of course, in the case of asymmetric data distributions the efficiency scores for the traditional model and for $\alpha = 1$ could be very different.

Except for values of α very high, the results of Table 5 show that any school in the sample could be technically efficient since the lower bound of the α-cuts, $\left(\tilde{\phi}_0^*\right)_\alpha^L$, takes value one. From values of $\alpha \geq 0.8$, some schools begin to look clearly inefficient. For example, school #117 presents confidence intervals for its efficiency score of $\left[\left(\tilde{\phi}_0^*\right)_{0.8}^L, \left(\tilde{\phi}_0^*\right)_{0.8}^U\right] = [1.05, 1.91]$, $\left[\left(\tilde{\phi}_0^*\right)_{0.9}^L, \left(\tilde{\phi}_0^*\right)_{0.9}^U\right] = [1.20, 1.75]$ and $\left[\left(\tilde{\phi}_0^*\right)_1^L, \left(\tilde{\phi}_0^*\right)_1^U\right] = [1.48, 1.48]$. This behavior of the results is related to the schools associated with great values for the traditional DEA efficiency score. In the case of the unit #117, it is the school with the greatest traditional score (1.47). Another example is unit #113, which has $\left[\left(\tilde{\phi}_0^*\right)_{0.9}^L,\left(\tilde{\phi}_0^*\right)_{0.9}^U\right]=[1.04,1.48]$ and $\left[\left(\tilde{\phi}_0^*\right)_1^L,\left(\tilde{\phi}_0^*\right)_1^U\right]=[1.29,1.29]$, while its traditional score is high (1.36).

Another interesting comparison is that associated with the increase of the number of fuzzy variables in the approach. What happens if we consider in our context only one fuzzy variable instead of three? Basically, we can conclude that the more variability there is in the data, the more differences there are in the results with respect to the traditional model, where all variables, inputs and outputs, are treated as crisp. To illustrate that, we show in Table 6 the results corresponding to the model solved assuming that only the variable MATH is fuzzy, while the other variables are crisp. Seeking brevity, we have also shown the results for some selected DMUs.

Note that, given any possibility level α, the more variables we consider in the model, the wider the confident intervals generated (α-cuts). Additionally, the more variables we consider, the smaller is the correlation between the traditional DEA score and the index (8). Particularly, the correlation associated with the model with one fuzzy variable (MATH) is very high (0.95) and the correlation of the model with three fuzzy variables is 0.76. Something similar happens with respect to the possibility of considering READING or ESCS as fuzzy variables.

<Please insert Table 6 approximately here>

Going back in detail to the results of Table 5, schools #18, #115 and #143 are efficient for any possibility level α. Given that the Kao and Liu model takes into account the intra-school variabilities in the data, we can be sure that these three schools are technically efficient in relation to the sample of units. It is worth mentioning that these three schools are also classified as efficient by the traditional DEA model. However, unit #119 is technically efficient under the classical perspective but clearly inefficient when the dispersion in the data distribution is incorporated to the analysis, since the upper bound of all the shown α-cuts are strictly greater than one. Additionally, the results for this particular unit must be affected by some asymmetry in its data since its efficiency score is 1.21 in the case of considering its modes for MATH, READING and ESCS, i.e. $\alpha=1$, instead of its means, which is therefore very far of the traditional DEA efficiency score ($\phi_0=1$).

The schools #84, #92, #99, #120 and #159 are fuzzy efficient exclusively for $\alpha=1$. For all other possibility levels, the upper bound of the α-cuts $\left[\left(\tilde{\phi}_0^*\right)_\alpha^L,\left(\tilde{\phi}_0^*\right)_\alpha^U\right]$ is greater than 1. For example, for unit #159, it is indicative that the overall index (8), the last column in Table 5, presents a value of 1.13, signaling as a summary that this school could improve their outputs (MATH and READING) by a 13% keeping the inputs constant. In contrast, the traditional radial DEA model suggests that this school performed efficiently and that it could serve as a peer for other American schools. For all these schools, the traditional radial model provides a value of one and, consequently, it classifies these units as technically efficient. However, considering the data dispersion through the fuzzy DEA model, this conclusion is not so obvious.

More in detail, school #115 is the unit that presents the highest variation coefficient in its fuzzy variables (16% for MATH and READING and 38% for ESCS) and, at the same time, it has a score of one for any possibility level α. Given that a high dispersion in the data is related, in general, with wider confidence intervals (α-cuts), the fact that the fuzzy efficiency score for this unit is one for any α could be interpreted as that this school is clearly technically efficient. On the other hand, school #119 presents the greatest difference between the traditional score and the index (8): 1.00 vs 1.30. In fact, for high possibility levels, as for example 0.8, the fuzzy efficiency score of this unit could be even 1.56, indicating that this school is probably technically inefficiency. However, note that, based on the traditional approach, the school would be classified as efficient. This school presents a relatively high variation coefficient equals to 26% in the variable ESCS. This fact could be part of the justification of these results in the fuzzy efficiency score. Overall, big differences between the traditional efficiency score and (8) are related to medium-high intra-school variabilities.

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Figures

Figure 1. Example of membership function.

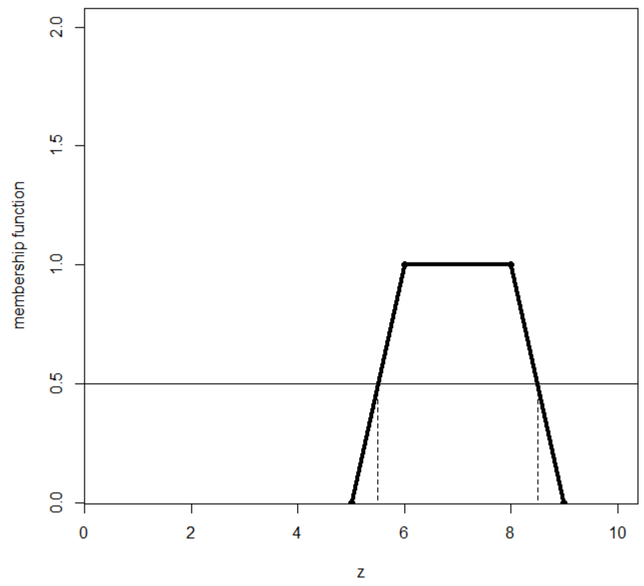
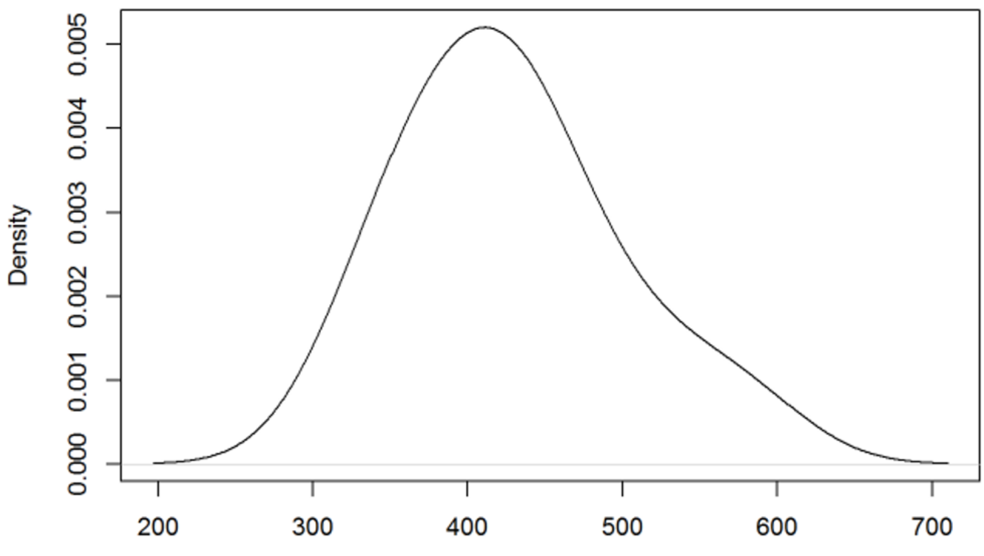


Figure 2. Example of a kernel.



Tables

Table 1: Descriptive statistics

Variable		Mean	Std. Dev.	Min.	Max.
Outputs (*Fuzzy)*					
	MATH	481.54	44.64	375.80	581.31
	READ	497.94	45.92	344.17	599.75
Inputs (*Fuzzy)*					
	ESCS	4.19	0.53	2.52	5.25
(Crisp)					
	SCMATEDU	4.59	0.53	2.92	5.65
	TEACHERS	6.57	2.78	0.85	30.82

Table 2: Descriptive statistics of the three fuzzy variables for school #115

	Mean	SD	Variation Coeff.
MATH	428.42	67.85	0.16
READ	435.69	71.37	0.16
ESCS	2.52	0.97	0.38

*n=28

Table 3: Skewness coefficient

Variable		Mean	SD
Fuzzy Outputs			
	MATH	0.12	0.39
	READ	-0.04	0.35
Fuzzy Input			
	ESCS	-0.25	0.41

Table 4: α-cuts for school #115

	MATH		READ		ESCS	
α	**$MATH^L$**	**$MATH^U$**	**$READ^L$**	**$READ^U$**	**$ESCS^L$**	**$ESCS^U$**
0.5	327.3703	499.5733	323.7687	538.5629	1.0634	3.9620
0.6	338.4127	484.8773	335.6648	516.7772	1.2406	3.7102
0.7	349.8246	471.6831	347.4921	493.8609	1.4169	3.4618
0.8	362.3308	458.5234	360.0707	471.4516	1.6053	3.2077
0.9	377.4797	443.5027	375.0882	449.0764	1.8318	2.9230
1	410.7212	410.7212	409.0850	409.0850	2.3516	2.3516

Table 5: Results for the traditional radial DEA model and the FDEA model

for different α values considering three fuzzy variables

		α-cuts												
	Trad. Model	0.5		0.6		0.7		0.8		0.9		1		
School	**Score**	**E^L**	**E^U**	**E^L**	**E^U**	**E^L**	**E^U**	**E^L**	**E^U**	**E^L**	**E^U**	**E^L**	**E^U**	**I_j**
1	1.30	1	1.88	1	1.78	1	1.68	1	1.58	1	1.46	1.26	1.26	1.21
2	1.09	1	1.70	1	1.58	1	1.47	1	1.36	1	1.25	1.05	1.05	1.17
3	1.14	1	1.61	1	1.53	1	1.45	1	1.38	1	1.31	1.12	1.12	1.15
4	1.18	1	1.95	1	1.82	1	1.68	1	1.55	1	1.40	1.17	1.17	1.23
5	1.08	1	1.61	1	1.54	1	1.46	1	1.38	1	1.29	1.09	1.09	1.14
6	1.24	1	2.13	1	1.99	1	1.86	1	1.73	1	1.59	1.27	1.27	1.27
7	1.04	1	1.49	1	1.41	1	1.34	1	1.27	1	1.20	1.03	1.03	1.12
9	1.19	1	2.05	1	1.87	1	1.71	1	1.56	1	1.42	1.13	1.13	1.26
10	1.18	1	1.90	1	1.76	1	1.63	1	1.51	1	1.39	1.14	1.14	1.22
12	1.11	1	1.55	1	1.46	1	1.38	1	1.31	1	1.23	1.08	1.08	1.15
13	1.14	1	1.78	1	1.66	1	1.56	1	1.46	1	1.36	1.15	1.15	1.19
14	1.12	1	1.81	1	1.66	1	1.53	1	1.40	1	1.28	1.07	1.07	1.19
16	1.19	1	1.89	1	1.79	1	1.68	1	1.58	1	1.46	1.26	1.26	1.21
17	1.11	1	1.65	1	1.54	1	1.44	1	1.34	1	1.24	1.03	1.03	1.18
18	1.00	1	1.00	1	1.00	1	1.00	1	1.00	1	1.00	1.00	1.00	1.00
19	1.27	1	2.04	1	1.91	1	1.78	1	1.65	1.05	1.53	1.30	1.30	1.25
20	1.07	1	1.66	1	1.57	1	1.48	1	1.40	1	1.29	1.06	1.06	1.16
21	1.10	1	1.64	1	1.53	1	1.44	1	1.35	1	1.26	1.07	1.07	1.16
22	1.05	1	1.53	1	1.43	1	1.35	1	1.27	1	1.18	1.00	1.00	1.13
23	1.08	1	1.56	1	1.48	1	1.40	1	1.32	1	1.24	1.04	1.04	1.14
24	1.09	1	1.62	1	1.53	1	1.45	1	1.37	1	1.29	1.12	1.12	1.15
26	1.19	1	2.05	1	1.92	1	1.80	1	1.67	1	1.53	1.28	1.28	1.26
27	1.08	1	1.88	1	1.72	1	1.58	1	1.45	1	1.29	1.02	1.02	1.22
28	1.10	1	1.61	1	1.54	1	1.47	1	1.39	1	1.30	1.12	1.12	1.14
29	1.16	1	1.83	1	1.73	1	1.62	1	1.51	1	1.36	1.06	1.06	1.21

	Trad. Model	α-cuts												
		0.5		0.6		0.7		0.8		0.9		1		
School	Score	E^L	E^U	E^L	E^U	E^L	E^U	E^L	E^U	E^L	E^U	E^L	E^U	I_j
30	1.18	1	1.84	1	1.73	1	1.63	1	1.51	1	1.31	1.08	1.08	1.19
31	1.16	1	1.86	1	1.75	1	1.65	1	1.53	1	1.41	1.22	1.22	1.21
32	1.12	1	1.66	1	1.56	1	1.47	1	1.38	1	1.28	1.09	1.09	1.16
33	1.04	1	1.53	1	1.42	1	1.32	1	1.23	1	1.16	1.01	1.01	1.13
34	1.14	1	1.68	1	1.57	1	1.47	1	1.38	1	1.29	1.10	1.10	1.18
35	1.05	1	1.61	1	1.51	1	1.39	1	1.29	1	1.17	1.00	1.00	1.15
36	1.07	1	1.63	1	1.51	1	1.39	1	1.29	1	1.20	1.00	1.00	1.15
37	1.08	1	1.71	1	1.63	1	1.54	1	1.45	1	1.34	1.08	1.08	1.17
38	1.02	1	1.63	1	1.54	1	1.43	1	1.32	1	1.22	1.02	1.02	1.14
40	1.00	1	1.50	1	1.44	1	1.38	1	1.32	1	1.23	1.05	1.05	1.12
41	1.04	1	1.58	1	1.50	1	1.41	1	1.33	1	1.23	1.05	1.05	1.13
43	1.11	1	1.81	1	1.67	1	1.54	1	1.43	1	1.32	1.06	1.06	1.20
44	1.04	1	1.52	1	1.44	1	1.36	1	1.29	1	1.21	1.04	1.04	1.13
45	1.13	1	1.62	1	1.54	1	1.47	1	1.40	1	1.32	1.17	1.17	1.15
47	1.11	1	1.62	1	1.53	1	1.44	1	1.36	1	1.27	1.11	1.11	1.15
48	1.14	1	1.64	1	1.55	1	1.47	1	1.39	1	1.30	1.12	1.12	1.18
49	1.07	1	1.63	1	1.52	1	1.43	1	1.34	1	1.24	1.06	1.06	1.15
50	1.06	1	1.71	1	1.60	1	1.49	1	1.39	1	1.28	1.09	1.09	1.17
51	1.11	1	1.67	1	1.56	1	1.46	1	1.34	1	1.25	1.08	1.08	1.17
52	1.14	1	1.77	1	1.63	1	1.50	1	1.37	1	1.26	1.08	1.08	1.19
53	1.13	1	1.83	1	1.69	1	1.57	1	1.45	1	1.33	1.12	1.12	1.21
54	1.13	1	1.73	1	1.63	1	1.54	1	1.45	1	1.34	1.09	1.09	1.17
55	1.29	1	1.98	1	1.86	1	1.73	1	1.60	1	1.47	1.25	1.25	1.25
56	1.12	1	1.75	1	1.66	1	1.57	1	1.47	1	1.36	1.19	1.19	1.18
57	1.11	1	1.84	1	1.70	1	1.57	1	1.43	1	1.28	1.05	1.05	1.20
58	1.09	1	1.99	1	1.85	1	1.72	1	1.60	1	1.47	1.23	1.23	1.24
59	1.22	1	1.89	1	1.77	1	1.66	1	1.55	1	1.43	1.17	1.17	1.22
60	1.06	1	1.58	1	1.47	1	1.37	1	1.29	1	1.21	1.05	1.05	1.16
61	1.06	1	1.54	1	1.44	1	1.34	1	1.25	1	1.17	1.02	1.02	1.13
62	1.18	1	1.89	1	1.73	1	1.60	1	1.48	1	1.36	1.14	1.14	1.23
63	1.15	1	1.72	1	1.61	1	1.51	1	1.42	1	1.34	1.18	1.18	1.18
65	1.19	1	2.01	1	1.82	1	1.66	1	1.52	1	1.38	1.16	1.16	1.25
66	1.20	1	1.89	1	1.76	1	1.64	1	1.52	1	1.37	1.12	1.12	1.21
67	1.19	1	1.91	1	1.77	1	1.64	1	1.52	1	1.40	1.16	1.16	1.22
68	1.02	1	1.52	1	1.44	1	1.37	1	1.30	1	1.21	1.03	1.03	1.12
69	1.25	1	2.12	1	1.97	1	1.83	1	1.70	1	1.54	1.26	1.26	1.27
70	1.16	1	1.81	1	1.69	1	1.56	1	1.44	1	1.33	1.14	1.14	1.19
71	1.16	1	2.03	1	1.87	1	1.72	1	1.57	1	1.41	1.17	1.17	1.25
72	1.14	1	1.65	1	1.52	1	1.43	1	1.34	1	1.25	1.09	1.09	1.17
73	1.11	1	1.55	1	1.46	1	1.38	1	1.30	1	1.22	1.05	1.05	1.14
74	1.14	1	1.87	1	1.73	1	1.61	1	1.49	1	1.37	1.13	1.13	1.21
75	1.11	1	1.63	1	1.53	1	1.43	1	1.34	1	1.26	1.09	1.09	1.16

		α-cuts												
	Trad. Model	0.5		0.6		0.7		0.8		0.9		1		
School	Score	E^L	E^U	E^L	E^U	E^L	E^U	E^L	E^U	E^L	E^U	E^L	E^U	I_j
76	1.13	1	1.74	1	1.63	1	1.53	1	1.42	1	1.31	1.08	1.08	1.19
77	1.18	1	1.66	1	1.54	1	1.45	1	1.36	1	1.29	1.13	1.13	1.16
78	1.24	1	1.81	1	1.71	1	1.62	1	1.53	1	1.42	1.22	1.22	1.19
79	1.08	1	1.85	1	1.71	1	1.60	1	1.48	1	1.35	1.09	1.09	1.21
80	1.18	1	2.11	1	1.94	1	1.77	1	1.61	1	1.46	1.19	1.19	1.26
81	1.07	1	1.71	1	1.58	1	1.48	1	1.38	1	1.29	1.10	1.10	1.19
82	1.14	1	1.87	1	1.74	1	1.62	1	1.50	1	1.37	1.13	1.13	1.21
83	1.18	1	1.79	1	1.69	1	1.60	1	1.51	1	1.39	1.16	1.16	1.21
84	1.00	1	1.38	1	1.32	1	1.26	1	1.20	1	1.13	1.00	1.00	1.09
85	1.26	1	2.05	1	1.92	1	1.79	1	1.66	1	1.51	1.24	1.24	1.24
87	1.05	1	1.56	1	1.49	1	1.39	1	1.29	1	1.19	1.00	1.00	1.13
88	1.15	1	1.79	1	1.67	1	1.57	1	1.48	1	1.38	1.14	1.14	1.19
89	1.20	1	1.95	1	1.83	1	1.70	1	1.56	1.01	1.42	1.18	1.18	1.23
91	1.11	1	1.79	1	1.68	1	1.59	1	1.46	1	1.33	1.09	1.09	1.18
92	1.00	1	1.43	1	1.36	1	1.29	1	1.22	1	1.14	1.00	1.00	1.10
93	1.04	1	1.61	1	1.50	1	1.40	1	1.31	1	1.21	1.02	1.02	1.15
94	1.12	1	1.98	1	1.84	1	1.71	1	1.58	1	1.44	1.16	1.16	1.24
95	1.05	1	1.71	1	1.60	1	1.50	1	1.39	1	1.28	1.06	1.06	1.17
96	1.07	1	1.52	1	1.45	1	1.38	1	1.31	1	1.23	1.08	1.08	1.12
97	1.15	1	1.90	1	1.77	1	1.65	1	1.53	1	1.40	1.15	1.15	1.21
98	1.10	1	1.81	1	1.69	1	1.58	1	1.46	1	1.33	1.07	1.07	1.20
99	1.00	1	1.48	1	1.38	1	1.29	1	1.21	1	1.13	1.00	1.00	1.12
100	1.16	1	1.77	1	1.66	1	1.56	1	1.46	1	1.37	1.17	1.17	1.20
101	1.27	1	2.09	1	1.94	1	1.81	1	1.67	1	1.52	1.24	1.24	1.27
102	1.11	1	1.70	1	1.61	1	1.52	1	1.43	1	1.34	1.14	1.14	1.16
103	1.11	1	1.79	1	1.67	1	1.54	1	1.40	1	1.27	1.07	1.07	1.18
105	1.08	1	1.62	1	1.54	1	1.46	1	1.38	1	1.30	1.12	1.12	1.15
106	1.07	1	1.54	1	1.46	1	1.38	1	1.30	1	1.21	1.04	1.04	1.13
107	1.13	1	1.62	1	1.53	1	1.45	1	1.36	1	1.26	1.10	1.10	1.15
108	1.04	1	1.56	1	1.48	1	1.39	1	1.30	1	1.21	1.03	1.03	1.13
109	1.14	1	1.77	1	1.66	1	1.55	1	1.42	1	1.31	1.11	1.11	1.19
110	1.10	1	1.70	1	1.62	1	1.53	1	1.45	1	1.36	1.14	1.14	1.17
111	1.18	1	1.97	1	1.82	1	1.68	1	1.55	1	1.41	1.17	1.17	1.24
112	1.24	1	1.91	1	1.77	1	1.65	1	1.53	1	1.43	1.24	1.24	1.22
113	1.36	1	1.93	1	1.81	1	1.70	1	1.59	1.04	1.48	1.29	1.29	1.26
114	1.10	1	1.85	1	1.71	1	1.57	1	1.45	1	1.32	1.00	1.00	1.20
115	1.00	1	1.00	1	1.00	1	1.00	1	1.00	1	1.00	1.00	1.00	1.00
116	1.06	1	1.47	1	1.41	1	1.35	1	1.29	1	1.22	1.07	1.07	1.11
117	1.47	1	2.36	1	2.21	1	2.06	1.05	1.91	1.20	1.75	1.48	1.48	1.33
118	1.02	1	1.56	1	1.49	1	1.41	1	1.33	1	1.23	1.04	1.04	1.13
119	1.00	1	1.97	1	1.82	1	1.68	1	1.56	1	1.44	1.21	1.21	1.30
120	1.00	1	1.35	1	1.28	1	1.22	1	1.16	1	1.09	1.00	1.00	1.09

		α-cuts												
	Trad. Model	0.5		0.6		0.7		0.8		0.9		1		
School	Score	E^L	E^U	E^L	E^U	E^L	E^U	E^L	E^U	E^L	E^U	E^L	E^U	I_j
121	1.24	1	1.88	1	1.73	1	1.61	1	1.51	1	1.40	1.16	1.16	1.28
122	1.15	1	1.71	1	1.59	1	1.48	1	1.40	1	1.31	1.13	1.13	1.19
123	1.13	1	1.67	1	1.58	1	1.49	1	1.41	1	1.32	1.12	1.12	1.16
124	1.10	1	1.79	1	1.68	1	1.58	1	1.49	1	1.39	1.16	1.16	1.19
126	1.10	1	1.58	1	1.49	1	1.41	1	1.33	1	1.25	1.02	1.02	1.15
127	1.03	1	1.47	1	1.39	1	1.32	1	1.25	1	1.17	1.01	1.01	1.12
129	1.09	1	1.60	1	1.49	1	1.39	1	1.30	1	1.21	1.04	1.04	1.15
130	1.14	1	1.71	1	1.63	1	1.55	1	1.47	1	1.37	1.14	1.14	1.17
131	1.11	1	1.74	1	1.62	1	1.51	1	1.40	1	1.29	1.09	1.09	1.18
132	1.15	1	1.92	1	1.75	1	1.62	1	1.51	1	1.38	1.17	1.17	1.22
133	1.14	1	1.82	1	1.71	1	1.62	1	1.52	1	1.41	1.20	1.20	1.20
134	1.07	1	1.64	1	1.52	1	1.42	1	1.33	1	1.23	1.04	1.04	1.16
135	1.08	1	1.74	1	1.61	1	1.50	1	1.39	1	1.26	1.03	1.03	1.18
136	1.06	1	1.50	1	1.42	1	1.33	1	1.25	1	1.16	1.00	1.00	1.12
137	1.03	1	1.65	1	1.55	1	1.45	1	1.36	1	1.24	1.00	1.00	1.16
138	1.12	1	1.52	1	1.45	1	1.38	1	1.32	1	1.25	1.06	1.06	1.14
139	1.12	1	1.56	1	1.50	1	1.43	1	1.37	1	1.30	1.15	1.15	1.13
140	1.08	1	1.57	1	1.49	1	1.41	1	1.34	1	1.26	1.10	1.10	1.13
141	1.06	1	1.66	1	1.55	1	1.44	1	1.33	1	1.21	1.00	1.00	1.17
142	1.18	1	1.63	1	1.55	1	1.47	1	1.40	1	1.32	1.17	1.17	1.15
143	1.00	1	1.00	1	1.00	1	1.00	1	1.00	1	1.00	1.00	1.00	1.00
145	1.06	1	1.80	1	1.69	1	1.57	1	1.45	1	1.33	1.06	1.06	1.19
146	1.16	1	1.82	1	1.70	1	1.59	1	1.48	1	1.36	1.13	1.13	1.21
147	1.14	1	1.68	1	1.60	1	1.52	1	1.45	1	1.38	1.21	1.21	1.16
148	1.23	1	1.92	1	1.82	1	1.73	1	1.63	1	1.51	1.25	1.25	1.21
149	1.05	1	1.43	1	1.35	1	1.28	1	1.22	1	1.16	1.03	1.03	1.12
150	1.08	1	1.55	1	1.45	1	1.37	1	1.28	1	1.20	1.03	1.03	1.14
151	1.23	1	1.69	1	1.60	1	1.51	1	1.42	1	1.35	1.21	1.21	1.18
152	1.34	1	2.10	1	1.99	1	1.88	1	1.76	1	1.62	1.37	1.37	1.26
153	1.10	1	1.58	1	1.52	1	1.45	1	1.39	1	1.32	1.16	1.16	1.14
154	1.17	1	1.84	1	1.70	1	1.57	1	1.45	1	1.34	1.13	1.13	1.20
155	1.10	1	1.73	1	1.63	1	1.53	1	1.42	1	1.31	1.03	1.03	1.17
157	1.23	1	1.91	1	1.76	1	1.63	1	1.51	1	1.41	1.19	1.19	1.22
158	1.15	1	1.83	1	1.71	1	1.60	1	1.50	1	1.38	1.16	1.16	1.20
159	1.00	1	1.52	1	1.44	1	1.37	1	1.29	1	1.18	1.00	1.00	1.13
161	1.14	1	1.88	1	1.76	1	1.62	1	1.48	1	1.35	1.11	1.11	1.21

Table 6: Results for the traditional radial DEA model and the FDEA model

for different α values considering only one fuzzy variable (MATH). Some selected schools

		α-cuts												
	Trad. model	0.5		0.6		0.7		0.8		0.9		1		
School	Score	E^L	E^U	E^L	E^U	E^L	E^U	E^L	E^U	E^L	E^U	E^L	E^U	I_0
18	1.00	1	1.00	1	1.00	1	1.00	1	1.00	1	1.00	1.00	1.00	1.00
19	1.27	1	1.27	1	1.27	1.04	1.27	1.09	1.27	1.16	1.27	1.27	1.27	1.59
84	1.00	1	1.01	1	1.01	1	1.01	1	1.01	1	1.01	1.00	1.00	1.01
89	1.20	1	1.25	1	1.25	1	1.25	1.00	1.25	1.07	1.25	1.24	1.24	1.53
92	1.00	1	1.00	1	1.00	1	1.00	1	1.00	1	1.00	1.00	1.00	1.00
99	1.00	1	1.00	1	1.00	1	1.00	1	1.00	1.00	1.00	1.00	1.00	1.00
111	1.18	1	1.29	1	1.29	1	1.29	1	1.29	1	1.29	1.17	1.17	1.57
113	1.36	1	1.51	1	1.51	1.02	1.51	1.07	1.51	1.13	1.47	1.29	1.29	2.05
115	1.00	1	1.00	1	1.00	1	1.00	1	1.00	1	1.00	1.00	1.00	1.00
117	1.47	1.07	1.47	1.13	1.47	1.18	1.47	1.24	1.47	1.31	1.47	1.45	1.45	2.26
119	1.00	1	1.08	1	1.08	1	1.08	1	1.08	1.05	1.08	1.08	1.08	1.17
120	1.00	1	1.00	1	1.00	1	1.00	1	1.00	1	1.00	1.00	1.00	1.00
143	1.00	1	1.00	1	1.00	1	1.00	1	1.00	1	1.00	1.00	1.00	1.00
159	1.00	1	1.03	1	1.03	1	1.03	1	1.03	1	1.03	1.00	1.00	1.05